Comparative Geometry in Distance Education

I Lénárt

Faculty of Natural Sciences, Mathematics Teaching and Education Centre, Eötvös Loránd University, H-1117 Pázmány Péter sétány 1/C, Budapest, Hungary

E-mail: lenarti@cs.elte.hu

Abstract. I have been working on an educational project called Comparative Geometry for decades. The project is based on teaching and learning plane geometry and spherical geometry simultaneously, mainly through direct experimentation with hands-on tools, and intensive use of discussion between classmates. This work has gravely been affected by the changes that occurred two months ago because of the pandemic, still causing emergencies in many areas of education. In this article, I describe how I tried to adapt the work of a university course to an emergency; the methods by which I enabled direct experimentation and personal communication that were not possible in the given situation; my efforts to reduce the drawbacks of the situation and take advantage of the potential benefits.

1. Introduction

The fundamental idea of the Comparative Geometry project is to teach and learn two (or later more) different systems of geometry simultaneously. Students compare and contrast concepts and theorems in two or more different worlds of geometry, as in the book of Henderson and Taimina for university students [1], or Lénárt's book for the upper elementary and secondary school [2], or the book of Rybak and Lénárt for the interested layman and the professional teacher as well [3].

Comparative Geometry starts with the Euclidean geometry of the plane and the Menelaosian geometry of the spherical surface. Both of them have their origins in the antiquity.

However, in the past two millennia Euclidean geometry has maintained absolute dominance in the European cultural and educational tradition. Spherical geometry had been involved in secondary education until the middle of the twentieth century, because of its applications in geography, navigation, astronomy and art, but then gradually was pushed out of the school curriculum. Only in recent years has it returned in a completely subordinate role to plane geometry and in science popularization books such as VanBrummelen's work [4].

As for hyperbolic geometry, it has only recently earned an aside in the secondary curriculum, partly due to its role in modern physics, partly as an example of "imagining the unimaginable".According to Alexandrov, "Lobachevskian geometry can hardly be included in secondary school curricula, but it seems essential to give pupils an idea of it and to show them the greatness of the human spirit, capable of creating unimaginable concepts and theories which, in the course of time, proved to be comprehensible and fruitful." [5] A Hungarian poet wrote about Bolyai, one of the discoverers of hyperbolic geometry: "...I kidnapped the treasure of the unimaginable, and laugh at you, old Euclid, captive of your own laws."[6]

To this date, spherical geometry is just as strange and unimaginable for the vast majority of students as hyperbolic geometry used to be for scientists of the first half of the 19th century.

In Hungary, teaching about Comparative Geometry in colleges and universities started three decades ago at ELTE University, Budapest, in the form of an elective course, under the name “Ball geometry”. Participants are prospective teachers in primary and secondary schools. During the years many future kindergarten teachers have also attended, together with many Erasmus students from different parts of the world.

Fig. 1. Erasmus students experimenting with spherical construction materials.

Fig. 2. Presentation for 350 pupils in Copernicus Science Centre, Warsaw, with Anna Rybak (University of Białystok, Poland.)

2. Syllabus of the course

One semester consists of about 10-11 lessons, 90 minutes each.

The material of the course has continuously been changing, but it has boiled down by now to the following list of topics:

- Basic idea and goals of comparative geometry on plane and sphere. (The ideology behind the educational project.)
- Freehand drawing on the sphere, with students' spontaneous insights about important concepts of spherical geometry that should be studied later during the course.
- Point and straight line on plane and sphere. (Perhaps it is the most difficult transition for the student to accept a new interpretation of the term "straight".) Parallels and perpendiculars.
- Distance on plane and sphere. Circles on plane and sphere. Concentric circles. Point, opposite point and equator.
- The draw-on globe: a blank map of the earth that can be drawn and wiped off, to estimate distances and areas, create illustrations to geography, history, environment, transportation, etc..
- The concept and measurement of angle on sphere and plane. (Another crucial topic that is easier to approach first on the finite sphere, only then turn to the infinite plane.)
- Polygons on plane and sphere. Biangle (or digon or lune) which is one of the big surprises for the beginner.
- Triangle and Euler triangle. Sum of sides and sum of interior angles.
- Congruence of triangles: side-side-side, side-angle-side, angle-side-angle. (Essentially the same on plane and sphere)
- Congruence of triangles: angle-angle-side. (Stark difference between plane and sphere.)
- Congruence of triangles: side- side-angle. (Essentially thesame on plane and sphere.)
- Congruence of triangles: angle-angle-angle (Stark difference between plane and sphere.)
- Quadrilaterals on plane and sphere.(Trapezium, parallelogram, rhomb, rectangle, deltoid, square.)
- Measurement of area on the plane and on the sphere.

3. Background and methodology

As can be seen on the pictures above, the entire project is built on direct experimentation with hands-on natural and artificial tools. In addition, personal communication, friendly atmosphere, the right to err and to correct errors on the side of the student and (surprisingly) of the teacher is of vital importance in the course.

Before the pandemic, whenever it was possible, I have always tried to inspire students to conduct independent research, free and informal conversation in the classroom, and even a lively debate on the concept at issue. Manipulative tools were extremely helpful to reduce the leading position of the lecturer and the percentage of frontal presentation. I often hid out of sight of students, sitting in the back row of the lecture hall to help students perceive their own central role, appreciate their own results, instead of expecting the ultimate truth from the lecturer.

Another important factor was the extremely diverse knowledge of the participants, especially those arriving from another country with different curricula in mathematics. Some students admitted that they had never heard of Euclid, while others were well versed in Euclidean geometry. There was one point where everyone agreed: They had almost no knowledge about spherical geometry, let alone other geometries.

The situation changed dramatically due to the pandemic which hit the Hungarian educational network in mid-March 2020. Teachers and students had to switch to distance learning overnight, which was a largely unknown, unusual way of communication.It is worth noting that I myself participated in my first Zoom conference during these weeks.

The sudden change had a serious impact on the Ball geometry course, a total of 29 students, including 5 Erasmus students from Spain, Italy, Switzerland and Iran. The direct experimentation with hands-on tools, discussion and debate among classmates became very difficult in the unusual circumstances of distance learning.

In this emergency, the program was changed as follows: Each week, students received the elaboration of the next topic, with questions and related tasks. The received material had to be returned with students' own comments, questions, solutions, to which I answered again, with the necessary explanations and corrections. With this indirect communication, I tried to make up for the personal encounter with the students. These elaborations were not graded.

Grading and evaluation took place at the final exam. Before the pandemic, it took a mixed form of oral and written communication in the classroom. This procedure was not feasible in an emergency situation. Instead, each student received a topic that she worked out on 1-3 pages. In addition, I put together a collection of 17 exercises from which each student solved three tasks of her own choice. (I made sure it wasn't possible to pick three that were too easy to solve.)

Students were expected to send their work to me. I commented on the tasks, but did not fully solved them (as seen below), and offered a mark that could be changed if the candidate returned an improved work.

4. Sample topic sent to the students weekly

Following is a part of the topic sent to the students about "Angle and angle measurement". (Pictures are without numbering and captions here, because the accompanying text explains the picture.)Hopefully this detail illustrates the content and style of the descriptions.

I reiterate that the text is also intended for those who may have difficulty interpreting the basic concepts of Euclidean geometry.

Angle and angle measurement

Table 1. Summary of what we have done before

	Plane	**Sphere**
Simplest element Simplest line	Point Straight line	Point Spherical straight line / Great circle = greatest circle on the sphere that divides it into two hemispheres 
How to draw a simplest line	Planar straightedge	Scaled edges of spherical ruler on plastic sphere / tait string or rubber band on an orange  
Distance of two points	Length of straight segment between points ;	 Length of minor (shorter) arc of great circle between not opposite points  any meridian (half of a great circle) between opposite points
Unit of distance	Arbitrary (finite) segment	Can be a full circumference (equator), but we choose 1/360 of it, called degree. Each full great circle 360°, the opposite points 180°.  No distiction made

		180°. between smaller or greater spheres
Circle	Locus (geometric location) of equidistant points from a center on the plane	Locus (geometric location) of equidistant points from a centre on the sphere. The opposite point of the centre is also a centre.
Concentric circles with the same center	No circle among them which is a straight line. No two different circles with equal length of perimeter.	One circle of radius 0° / one circle of radius 180° (opposite points). Circle of radius 90° is great circle/straight line. Each circle of not 90° radius has a mate of same length of perimeter (Tropic of Cancer / Tropic of Capricorn) 
Pole and equator (also called pole and polar)	Similar correspondence is also possible between points and straight lines on the plane, but much more difficult than on the sphere.	Each spherical point has an opposite point and an equator. Each great circle/spherical straight line has two opposite pole points. has an opposite point and an equator. 

Important request:

Following are questions, explanations and exercises. Answers and solutions to the questions are in the text after the given question.

DO NOT READ FURTHER

the text to the answer/solution before you think it over and write your opinion. This is the best way to really understand it, so please do not spoil the game!

Concept of angle

This is one of the most difficult and most obscure concepts of elementary geometry not only for kids, but also for university students and even practising teachers – why?

Typically, the concept of measuring geometric objects begins with measuring distance, angle, and area. In plane geometry, the unit of distance is a segment, a piece of a finite line. Likewise, the unit of area (as we will see) is a finite unit square. In contrast, the unit of the angle region on the plane is an infinite region that cannot be fully displayed on a sheet of paper or a computer screen. However, the child (and the adult) only sees the finite region and derives its properties based on what he actually perceives on the image.

Question 1: What is an angle?

"An angle is a circle that we draw in the corner of the angle." Nonsense! If the angle is actually a circle, why should we look at the angle separately? Also, where exactly is the corner of the angle where we put the circle?

READ FURTHER ONLY AFTER WRITING YOUR OPINION!

Given a flat surface or a spherical surface, we select a point on it, and draw two segments/arcs from that point. We extend them until the surface is divided into two separate regions.

On the plane, we have to extend the two segments into two infinite rays in both directions. Should we stop at a finite point, we only cut into the infinite sheet of paper, but did not split it into two separate regions.

Question 2: How far should we extend the two arcs on the sphere?

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Luckily, we don't have to go to infinity, but only to the opposite point where the two spherical straight lines meet, and the surface is decomposed into two spherical regions.

Question 3: How much larger is the area of a 90º angle region on the plane than that of a 30º angle region?

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They have no measure of area at all! The area of both domains is infinite and immeasurable, just as we cannot compare the lengths of two half-lines. Therefore, angle measurement on the plane does not compare the measure of area of angle regions. If we say, “A 90º angle region is three times as large as the 30º angle region,” this means that three 30º plane angle regions can be placed onto the 90º angle region without gaps and overlaps, but it says nothing about their measure of area.

Probably the main reason of the problems with the concept of angle is that we can compare two infinite angular regions on the plane so that we miraculously get a finite number out of this comparison.

As a possible way out of this dilemma, we suggest introducing the concept of an angle first on the sphere and then turning to the planar case. The finite spherical surface precedes the infinite flat surface which makes certain angle issues easier to understand and visualize, especially for children.

Angle region on the sphere

Question 4: Draw a point on the sphere and two arcs of great circles from this point. Extend the two arcs until they meet again. How long will they be? How many parts is the surface of the entire sphere divided into? Try the same with an orange by cutting a piece out of the peel!


Fig. 4. A biangle on the plastic sphere. Fig. 5. A biangle cut out from an orange peel.

READ FURTHER ONLY AFTER WRITING YOUR OPINION!

5. Some exercises developed by students and my answers

Students were definitely asked to not only give the final result, but to describe the steps leading to the solution.

Exercise 1: What is the sum of exterior angles in a spherical triangle?

Everyone who has chosen this problem has solved the planar case correctly, although the clarity of the description has changed from almost childish wording to mathematically correct notation. The spherical counterpart was much more difficult, due to the adherence to the Euclideanway of thinking, and orientation in another world of geometry.

Exercise 1 is the work of a prospective kindergarten teacher who is also deeply interested in art and literature. She recently finished her thesis about ballet choreography, but also showed deep interest in geometry, partly because of its connection with dance motifs.

“In the plane: If we take a regular triangle first in the plane, all three angles will be 60°. The supplementary angle adds 180 ° to the original angle, so all we need is to subtract 60° from 180 ° to get the number of additional angles. With this method, we see that the additional angle of one of the three interior angles will be 120° (180-60). The exercise asks about the sum of the supplementary angles of the plane triangle, so I multiply 120 by 3, since we have a total of three angles in the triangle. As a result, I obtained that the sum of the additional angles of a regular planar triangle is 360 °.

On a sphere, I tried to indicate the supplementary angles by drawing a regular triangle out of paper (with 4-4-4 cm sides) and then redrawing it on the sphere. (Fig. 6.) I stuck toothpicks in the tops to keep the rubber bands fixed on the surface.

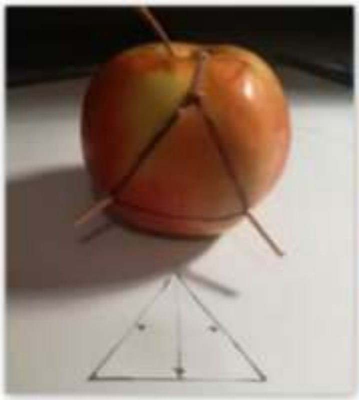

Fig. 6. Student's illustration to constructing a regular triangle on plane and sphere.

Fig. 7. Student's illustration to interior and exterior angles of a triangle.

Fig. 8. Student's illustration to smaller and bigger regular spherical triangles.

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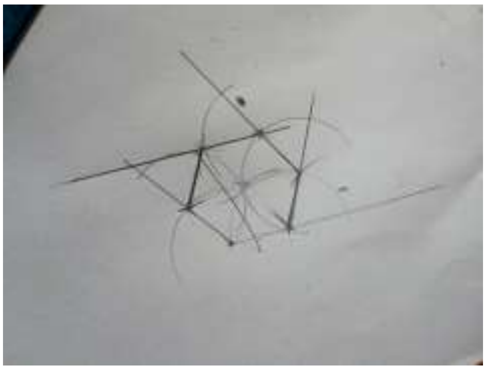
The rubber bands were used to lengthen the sides of the triangle, thus helping to draw supplementary angles, as seen on my second picture (Fig. 7.). If the original angle is supplemented by 180 for the supplementary angle, the procedure will be the same as for the plane triangle. The only difference, however, is that the internal angles are not so easy to calculate, as it is not a fixed number of the sum of the internal angles. Even because my triangle is straight and I know the lengths of the sides, I can't be sure of the number of internal angles, as their sum can range up to 540°. In this case, we are talking about a regular triangle, where I think the inner angle on the sphere is also 60 °, because I started from paper, so from a plane triangle. Thus, the sum of the additional angles is 360°. For regular triangles, where all interior angles are the same on the sphere, the sum of the additional angles is 360°, but this is the minimum value. I calculate the maximum by dividing 540 by 3, the 180 ° comes out, can this be the size of an angle at all, if it's just a straight line, a great circle on a sphere?! If I continue to use this number as a basis, I won't have a supplementary angle, i.e. it will be 0°, since I don't have to add 180 to anything to be 180. The sum of the maximum supplementary angles is therefore 0° (Fig. 8.)."

In the following exercises I give the full correspondence between the student and myself. The first message of the student and my answer are given in upright letters, the second exchange in italics, and the third exchange upright again.

Exercise 2: Each angle of a regular polygon is 120 degrees. How many sides does it have on the plane and on the sphere?

Student: Six sides on plane and sphere. First I constructed it on a sheet of paper, and I got six sides. Then I constructed it on a sphere, and again I got six sides, but the construction did not work out exactly, that's why one side is smaller than the others. (alpha=120^0)



It can also be derived from the sum of the interior angles of regular polygons, eg: triangle: 180°, square: 360°, pentagon: 540°, hexagon: 720°,… Then we multiply 120 by 3, 4, and so on. We are talking about a polygon that will be equal to that number.6*120= 720

Teacher: Oops! These angles belong to regular polygons on the PLANE – but what about the SPHERE? Maybe the construction on the apple was still correct?

Student: I tried again, but it is not possible to construct a regular polygon with all its angles 120°. I thought a lot (Teacher: You did it very well, that's the point!) about why it's not possible, but I didn't get an answer.

Teacher:Are you sure you can't? You started correctly by taking the sum of angles of a regular polygon, but you considered the polygons on the flat surface with fixed sum of angles. The picture is different on the sphere – remember? So the question is: Which regular polygons with 120° angles are possible on the sphere?

*Student: It can be a triangle, because 3*120 = 360 fits within the lower and upper limits of spherical triangles. It can be a quadrilateral, because 4*120 = 480 fits within the lower and upper limits of spherical quadrilaterals. It can be a pentagon: 5*120 = 600 fits within the limits of spherical pentagons. It can be a hexagon: 6*120 = 720 is the lower limit of the sum of angles in a spherical hexagon.*

*Teacher: Excellent! Two small comments: The spherical biangle is missing from your list. It is also a regular polygon and the angle 2*120° = 240° fits well within the limits- As for the hexagon, you were right and wrong at the same time. You correctly wrote that 6*120° = 720° is the lower limit, but such a hexagon only exists when it has degenerated to a spherical point. As soon as you leave the point to construct a larger regular hexagon on the sphere, the angles increase from 120° to 180°. Your construction on the apple was correct. The sixth side must be different!*

Exercise 3: How many ways to cut a spherical biangle into two congruent triangles?

Student: There is only one way for that. We cut the biangle along an axis of symmetry that is perpendicular to both semicircles. If we draw a line elsewhere that is right-angled on both semicircles, the two triangles will not be congruent, because one will be bigger than the other. If we *draw a non-perpendicular line between the two semicircles, we get two different triangles again. ($a_1=a_2$)*

Teacher: Are you sure? Have you tried this on a real biangle? Besides, can you draw two different great circles which are both perpendicular to both sides of the biangle?

Student: There are an infinite number of such lines, but these lines must intersect the two semicircles so as to form the same angles. The line on the figure intersects one of the semicircles at an angle labelled with one arc and another angle with two arcs. The same line intersects the other semicircle at two angles of the same measure respectively. Thus, two identical triangles are formed.

Teacher: Super! If you look at it more closely: Through which point of the spherical biangle does the intersecting line pass?

Student: The line must pass through the intersection of the two axes of symmetry, since the axes of symmetry cut the spherical biangle into two equal parts. And if the line passes through this point of intersection, the same triangle will be formed in both the vertical and horizontal directions.

Teacher: Faultless.

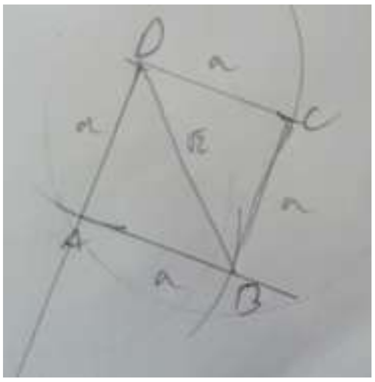
Exercise 4:Are there three points on the plane, any two of which have the same distance, so AB = BC = CA? And on the sphere?
Are there four points on the plane, any two of which have the same distance, so AB = CD = AC = BD = AD = BC? And on the sphere?

Student: There exist three points on the plane where the distance of any two is the same. These points give a regular triangle.

It is also possible to find such points on the sphere. Same is the regular triangle. The distance between all its sides and all three points is equal.

There are no four points on the plane with the same distance. The square is not appropriate either, because it has two distances that are longer than the others. / the diagonals are longer than the sides of the square.

Teacher: Excellent.



AB = CD = AD = BC = AC=BD

Student: There are no four such points on the sphere either. We can nicely prove this by taking 4 points that are two by two on a spherical line, and these two points are opposite points. The two lines intersect at right angles.

At first I thought that AB = CD = AC = BD = AD = BC. But on closer inspection and thought, I realized that this is not possible either, because the distance between the opposite points will be 180°, while between two adjacent points the distance will be 90°.

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Teacher: You are absolutely right about this, but can you reach your goal in another way? If you still have four such points on the sphere, then surely any three of these points give a regular triangle, as you correctly proved already. So take the three vertices of a regular spherical triangle and construct the noteworthy point (centre of gravity, orthocentre, etc.) in the triangle. Fix this point and change the three vertices of the regular triangle. Does anything come out of this?

Student: I constructed a regular triangle, that is, AB = BC = AC. I constructed the orthocentre, which is also the centre of gravity in a regular triangle. Thus I get that Am = Bm = Cm. BUT, in this case AB is not equal to Am because AB = AC = BC = 90°. Am = Bm = Cm = half of 90° = 45°. Thus, there are no four points, the distance of any two of which is the same.

Teacher: And if you left the point m where it is, but with the regular triangle go BEYOND the equator?

Student: We can find four such points on the sphere if the fourth point is not the orthocentre of the triangle, but its opposite point! There must be a position when the opposite point of the orthocentre is at equal distance from each vertex of the regular triangle.

Teacher: That's true! The four vertices of a regular tetrahedron designate four such points on the surface of the sphere which is circumscribed around the tetrahedron.

Dear Eszter, if I have any idea about math, THIS is math what you did during our correspondence. All your flawed and flawless thoughts and experiments showed the only – not royal - route to true knowledge. If you ever teach mathematics to your students, friends, or children, pass on the experience you just had. Good luck! Your exam mark is a top five!

Student: Dear teacher, I really enjoyed this new type of geometry. It was good to think about. Certainly it would have been even better personally in the classroom.

Exercise 5: In an isosceles triangle I found a right triangle. Then the triangle disappeared. What were the other two angles - on the plane and on the sphere?

Student: In an isosceles triangle the angles on the base are always equal. The triangle has two identical angles and one angle which may be different or also identical. Either the angle enclosed by the legs is a right angle, or the angle enclosed by the base and one leg is a right angle, but then the angle enclosed by the other leg and the base will also be a right angle (on the sphere, as with the meridians and the equator).

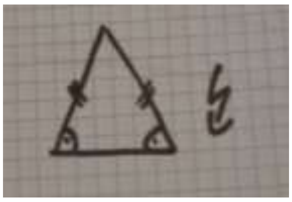
Teacher: Very good start!

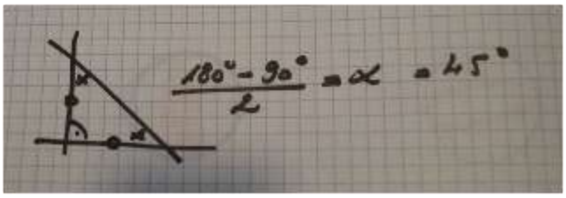
Student:

Plane:

- if the angle on the two bases is a right angle, then the triangle does not exist, because in a plane the interior angle of the triangles is always 180°, and 180° - (2*90°) = 0°.

- if the angle formed by the legs is a right angle, the other two angles are equal, so that they can be calculated: (180° -90°) / 2 = 45°. Then the angles of the triangle are: 90°, 45°, 45°.





Sphere:

- if the angle on the two bases is a right angle, then we are talking about a triangle whose base lies on the equator and whose vertex is defined by one of the poles. Then the angle enclosed by the legs can be anything between 0° and 180°, since no matter how much we can "open" this triangle Then the angles of the triangle are: 90°, 90°, 0° and 180°.

- if the angle enclosed by the legs is a right angle, then when we take the right angle, we get four equal, right-angled spherical biangles. These spherical biangles define isosceles triangles if we take the other two vertices of the triangle at equal distances from one of their vertices on both sides. Then the angles between the base and the legs will be equal and can be any size between 0 ° and 180 °. (Teacher: But why?) Then the angles of the triangle are: 90°, anything from 0° to 180°. (In this way we can also get a triangle with three right angles that cannot exist on the plane.)

Teacher: That's clear, I understand that. All you have to do is make sure that the angles can be of any size between 0° and 180°. Have you tried it on this beautiful round peach? Can it be 1° or 179°? If the green rubber gets very close to the intersection of the blue lines, do the two angles on the green rubber seem to be almost 0° or 180°?

Student:I was already thinking about this "between 0 and 180°" when I described the problem. When all the three angles are equal, it's 90 °, but then the angle increases in one direction, decreases in the other (by pushing the green rubber away). On the other hand, I realized that if the vertices were almost coinciding at one point, they look like are almost as a planar right-angled isosceles triangle, so in this case the magnitude of the angles is actually between 45° and 135°.

Teacher: Top five!

6. Conclusions

When drawing the conclusions, one must consider the extraordinary circumstances in which the communication between the students and the teacher took place.This experiment was not a carefully planned and executed research project including the appropriate control group for an analytic-statistical evaluation. An attempt was made to deal with an unforeseen emergency situation. The question was, Can the course be completed under such conditions? Is the online form of communication suitable for achieving the fundamental objectives of the course?

For that reason, I cannot supply relevant statistical analysis to the action research described above. Although a statistical test could have been performed between the grades of a previous and the current group, the results would be irrelevant due to the large difference between the initial conditions. Still, I give some numerical estimates regarding the current group, but these data are much more subjective than expected from the results of a planned research project.

Another interesting question is why I did not use any software to support the learning process online. One reason is that before the pandemic I did not have comparative geometry software that would fit the extraordinary circumstances of a sudden break. Besides, it would have been very difficult to organize a network overnight for students inside and outside the country. In addition, I had deeper objectives which I will describe later.

I was surprised by the perseverance of the students. Originally, there were 32 applicants, three of whom had already left the course at the beginning of the emergency. At the end of the distance learning, two students eventually did not complete the exam. One of them left without notice. The other student referred to family and financial problems, but would like to re-apply in the fall semester.

I definitely required the students not to copy-paste web pages instead of direct experiencing. There was only one student who quoted Internet sources far beyond the scope of the present syllabus, apparently without a real understanding. Of the 27 exam documents I received, 18 contained photos of self-made models, and an additional 5 contained self-made, carefully executed spherical diagrams drawn on sheets and photographed. 4 dissertations gave only written solutions without accompanying figures. (Of course, the addition of a drawing or photo was not mandatory. The point was to describe the route to the solution so that I could follow the reasoning with or without figures.)

What are the fundamental goals of the Comparative geometry project, and how were they fulfilled in distance learning?

The ideology behind the project was summarized in the article about Hungarian perspectives in mathematics education for the South African mathematics community [7].

One of the greatest problems in present-day mathematics teaching is the gap between inoperative and real knowledge that the student accumulates during the school years. Freudenthal wrote in his epoch-making paper almost fifty years ago: “Geometrical axiomatization cannot be meaningful as a teaching subject unless the student is allowed to perform these activities himself. Usually he is not allowed to do so.” [8].

I apologize for thinking the same about almost all topics in geometry teaching.

Is it necessary to define a geometric concept? The answer of the majority of the students is a definite no to the question. What is a straight line, a circle, a triangle, a square? No definition is required, they say. Just look at it and you will know the answer.

One of the harmful consequences of this perception is that the student does not understand the role of definition. An even more grave consequence is the student's belief that the true acquisition of knowledge is very different from the way you have to show in school. You have to write and memorize senseless and superfluous definitions in the school, while the real path to knowledge acquisition should be kept secret, in accordance with Brousseau’s didactic contract [9].

I was invited to give a geometry workshop for 17-year old high school students. When I entered the room, I addressed them in my old-fashioned manner: “Ladies and gentlemen, I am honoured to be here. My name is István Lénárt.” Then a girl in the front row asked: “Is this already must be written?” A similar experience can be obtained by posing questions such as: What arguments be made that a circle is a straight line on the sphere? Why is the South Pole also a centre of the Arctic Circle, in addition to the North Pole? Is there an obstacle to calling a regular spherical quadrilateral a square?

A question becomes a real challenge if and only if the same question arises on another surface, in another world of geometry. This idea is the basis of comparative geometry.

The same statement applies to the well-known problem of teaching about proof. To solve this problem, Freudenthal suggested the potential use of non-Euclidean geometry [8], which in fact means comparative geometry. This idea has been reinforced over the past few decades, as for example by Tall and others [10].

Another difficult problem is the adequate grading of the student. The viewpoint of comparative geometry contradicts the generally accepted method of evaluation. The mistake that testifies direct experimentation and independent thinking is much more valuable than a correct answer copied from a printed or online material. In Exercise 2, I highly appreciated the student’s fallacious conclusion from the figure on the apple, because it clearly showed the correct method of research: Dare to experiment, dare to draw conclusions on your own, dare to take the risk of error which can be corrected by further investigation. It is easier to correct a mistake through further research (as was the case with the apple above) than to get rid of the custom of unscrupulous takeover from uncontrolled sources.

Three decades of teaching experience suggested that direct experimentation and free informal communication were essential to achieving the goals of the comparative geometry project.

To what extent has this hypothesis been justified or refuted in the unforeseen situation caused by the pandemic? At the moment, I can only give a very subjective, partial answer to the question, which will certainly have to be reinforced by more accurate, statistically evaluated research projects.

I believe that the main goals to avoid the collapse of the course and to apply the basic forms of knowledge acquisition in the changed circumstances have been achieved.

Certainly, it would have been a great help if we could have organized the online communication for joint discussions and debates in Zoom conferences. This did not happen due to time and organization constraints and especially my own inexperience in this regard.

Still, in the correspondence with each student, I tried to maintain a relaxed tone reminiscent of peers’ conversations in the classroom. In translating the above exercises, I tried to illustrate this tone, although the language gap made it very difficult for me to convey the nuances of expression. Yet, according to the feedback from students, my efforts at least partially replaced the communication between the classmates.

It was very encouraging that direct experimental work proved to be successfully sustained under the changing circumstances. On the (more or less) spherical objects in the environment, students were able to create a system of geometry built on independent experience and observation. The most important result is that they appreciated and enjoyed this activity.

Beyond a certain level, precise spherical construction tools are of course important and even indispensable for the experiments. However, the extraordinary situation also proved that the first steps from Euclid to another system of geometry could be carried out with everyday objects and tools.Students can conduct introductory experiments in spherical geometry at school or at home even if special construction materials are not available. In addition, these commodities will continue to remind the student of the connection between mathematical abstraction and physical reality.

I have gained an equally important experience of the benefits of online communication with students in this project, both in terms of mid-year work and exams. Admittedly, in the past I have deliberately tried to avoid the dominance of online study over hands-on experimentation, because one of the main goals of the Comparative Geometry project is to restore the student's faith in her own senses and direct experience as opposed to the virtual, filtered reality through ICT sources.

After my current experience during the pandemic, I will try to incorporate online methods and communication in the semester and the exam period while still insisting on the priority of hands-on experimentation and direct personal discussion in the classroom. For example, online communication about the exam papers provides a more objective assessment that is acceptable to both the teacher and the student. This advantage can compensate for the additional time required for evaluation.

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